Nucleon and the Pion-Nucleon Trajectory. Determination of the Pion-Nucleon Coupling Constant*

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The low-energy p-wave pion-nucleon scattering amplitude is represented by a single Regge pole in the direct channel with correct analytical properties. The three parameters of the trajectory at threshold have been determined. From the residue of the pole at the nucleon position the πN coupling constant is determined to be $g^2=13.68$. The agreement at high energies is poor. A possible explanation of the discrepancy is that the analytically continued partial-wave amplitude at $J=\frac{1}{2}$ does not coincide with the physical amplitude. The difference is shown, from unitarity, to be approximately given at threshold by the contribution of an elementary nucleon pole. With such a term a good agreement is obtained up to 100 MeV with Woolcook's data and g^2 is found to be 13.03.

I. INTRODUCTION

RECENTLY, considerable attention has been given to the question whether the nucleon can be considered as a bound state of the π-N system lying on a Regge trajectory.[1–3] More precisely, whether the π-N, $I=\frac{1}{2}$ p-wave partial-wave amplitude analytically continued in angular momentum down to $J=\frac{1}{2}$ coincides with the physical amplitude and has the correct pole and the residue. This hypothesis may be checked through an analysis of the low-energy p-wave pion-nucleon scattering data because at these energies the amplitude is expected to be dominated by a single pole in the direct channel. We recall that a similar situation occurs in the case of n-p scattering where the low-energy-triplet scattering data can be fitted rather well with one Regge pole in the direct channel and, when extrapolated, does give the deuteron binding energy.[4] Some differences might be expected in the π-N case, because N is not a loosely bound state of πN system. Khuri and Udgaonkar[5] have recently computed the $I=\frac{1}{2}$ π-N phase shifts on the basis of a single linear trajectory passing through the nucleon and N^* ($J=\frac{5}{2}$, $I=\frac{1}{2}$). In the present calculation, we take into account the exact threshold behavior of the trajectory and the restrictions imposed by unitarity. It is shown that the single-pole hypothesis leads to some discrepancies. A possible explanation of these discrepancies is that the analytic continuation of the p-wave amplitude does not coincide with the physical amplitude at $J=\frac{1}{2}$, and that we have also the contribution of an elementary pole. The π-N coupling constant and the threshold parameters of the trajectory have been determined with and without an "elementary pole."

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II. THE PION-NUCLEON TRAJECTORY

If one assumes a Regge representation for the amplitude, the contribution of a single trajectory to the relevant p-wave amplitude $a_+^J(\nu)$, where $+$ indicates the amplitude with $l=J+\frac{1}{2}$, is given by[6]

$$a_+^{1/2}(\nu)=-\frac{1}{2}\frac{\beta(\nu)}{\alpha(\nu)-\frac{1}{2}}[e^{[\alpha(\nu)-1/2]\xi_1}+e^{[\alpha(\nu)-1/2]\xi_2}], \quad (1)$$

where

$$\xi_1=\cosh^{-1}\left(1+\frac{2\mu^2}{\nu}\right),$$

$$\xi_2=\cosh^{-1}\left[1+\frac{m^2}{2\nu}-\frac{(m^2-\mu^2)^2}{2\nu s}\right]; \quad (2)$$

ν is the square of the center-of-mass momentum, s the square of the center-of-mass energy, m and μ are the nucleon and the pion masses, $\alpha(\nu)$ the $I=\frac{1}{2}$ trajectory, and $\beta(\nu)$ its residue. The normalization of the amplitude is such that the elastic unitarity condition has the form

$$\mathrm{Im}a_+=(\nu/4\mu^2)^{1/2}a_+a_+^*.$$

The behavior of α and β, however, cannot be specified arbitrarily to be, for example, a straight line, because they are subject to restrictions imposed by unitarity. In particular, for small $|\nu|$, these quantities must have the form[7]

$$\alpha(\nu)=\alpha(0)-\frac{Y_\nu}{Y_J}\frac{\nu}{4\mu^2}+0(\nu^2)-\frac{1}{Y_J}\left(\frac{\nu}{4\mu^2}\right)^{\alpha(\nu)+1/2} \times\exp\{-i\pi[\alpha(\nu)+\frac{1}{2}]\}, \quad (3)$$

$$\beta(\nu)=\left(\frac{\nu}{4\mu^2}\right)^{\alpha(\nu)+1/2}b(\nu), \quad (4)$$

6 For the analytic continuation in J of the pion-nucleon amplitude see, for example, V. N. Gribov, Zh. Eksperim. i Teor. Fiz. **41**, 1962 (1961) [English transl.: Soviet Phys.—JETP **14**, 1395 (1962)]; V. Singh, Phys. Rev. **129**, 1889 (1963). We have also used the modification of the Watson transform giving the correct cut in z of the amplitude: N. Khuri, Phys. Rev. **130**, 429 (1963).
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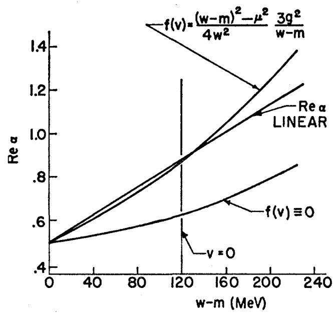

FIG. 1. The real part of the trajectory α is plotted against energy for two cases when the difference $f(\nu)$ between the analytically continued and the physical amplitude, [Eq. (23)], is zero or equal to the contribution of an elementary pole. For comparison the linear trajectory connecting N and N^* is also shown.

where Y_ν and Y_J are the real derivatives at $\nu=0$ of a real analytic function $Y(\nu,J)$ which has a right-hand cut beginning at the inelastic threshold. The quantity $b(\nu)$ will be assumed to be a slowly varying function of ν which is finite at $\nu=0$. We can then write in the neighborhood of threshold

$$b(\nu)\approx b(0)=\cos\pi[\alpha(0)-\tfrac{1}{2}]/Y_J\,, \tag{5}$$

so that the amplitude is finally given by

$$a_+^{1/2}(\nu)=\frac{\cos\pi[\alpha(0)-\frac{1}{2}]}{2Y_J[\alpha(\nu)-\frac{1}{2}]}\left(\frac{\nu}{4\mu^2}\right)^{\alpha(\nu)+1/2}\times[e^{[\alpha(\nu)-1/2]\xi_1}+e^{[\alpha(\nu)-1/2]\xi_2}]. \tag{6}$$

The three parameters determining $\alpha(\nu)$ and $\beta(\nu)$, namely $\alpha(0)$, Y_J and Y_ν, can be determined from the threshold behavior of the amplitude $a_+^{1/2}(\nu)$. The hypothesis of the dominance of a single trajectory is then checked by calculating $a_+^{1/2}(\nu)$ at higher energies and by computing the π-N coupling constant from the residue at the nucleon position. From Eq. (2) and the limit

$$\cosh^{-1}z\to\ln 2z+0(1/z)\,,$$

it is seen that the amplitude (6) has the correct p-wave threshold behavior. We take the following three relations to determine the three parameters

$$\left.\frac{a_+(\nu)}{\nu}\right|_{\nu=0}=-0.104\,,$$

$$\left.\frac{d}{d\nu}\frac{a_+(\nu)}{\nu}\right|_{\nu=0}=0.095\,,$$

in units of $\mu=1$ as given by Woolcook,[8] and the condition that the nucleon is on the trajectory, i.e., $\alpha(\nu_N)=\frac{1}{2}$, where

$$\nu_N=-\mu^2(1-\mu^2/4m^2)\,.$$

The calculation yields

$$\alpha(0)=0.125\,,\quad Y_J=-10.08\,,\quad Y_\nu=5.57\,,$$

and the residue of the pole at ν_N gives for the coupling constant

$$g^2=13.68\,.$$

The resulting curves for the trajectory and the amplitude are given in Figs. 1 and 2 as curves marked $f(\nu)=0$, with Woolcook's data and the linear $\alpha(\nu)$ between N and N^* for comparison. The choice of the condition $\alpha(\nu_N)=\frac{1}{2}$ is for convenience because of the trancendental nature of the form of $\alpha(\nu)$ in Eq. (3); instead of this condition an amplitude value near threshold, at, say, 10 MeV could have been taken with exactly the same result.

The results obtained in this way are perhaps not too satisfactory. The discrepancy at higher energies both in amplitude and in the expected behavior of the trajectory may be explained by other nearby singularities. Although it is possible that this is indeed the case we shall, however, suggest a second possibility.

III. EXTRAPOLATION TO THE POLE

We recall that in the relativistic theory the continuation in angular momentum is carried out from a dispersion relation of the type[9,10]

$$A(s,t)=\frac{g^2}{s-m^2}+C+\frac{1}{\pi}\int_{t_0}^{\infty}dt'A_t(s,t')\left[\frac{1}{t'-t}-\frac{1}{t'}\right], \tag{7}$$

where we have assumed, for definiteness, one bound state and one subtraction, and have omitted the exchange term. The physical amplitude $A_l(s)$ is given by the projection

$$A_l(s)=\frac{g^2}{s-m^2}\delta_{l0}+C\delta_{l0}+\frac{1}{\pi}\int_{t_0}^{\infty}dt'A_t(s,t')\times\left[\frac{1}{2\nu}Q_l\left(1+\frac{t'}{2\nu}\right)-\frac{\delta_{l0}}{t'}\right]. \tag{8}$$

However, as it stands, this expression cannot be used to

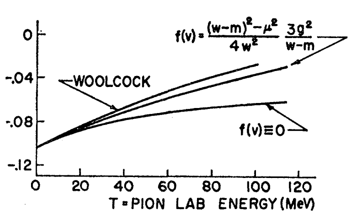

FIG. 2. The real part of the p-wave amplitude ($J=\frac{1}{2}$, $I=\frac{1}{2}$) as a function of energy for the two cases as in Fig. 1 compared to Woolcook's data.

[8] W. S. Woolcook, Ph.D. thesis, reproduced in J. Hamilton, P. Menotti, G. C. Oades, and L. L. J. Vick, Phys. Rev. **128**, 1881 (1962).

[9] V. N. Gribov, Zh. Eksperim. i Teor. Fiz. **41**, 1962 (1961) [English transl.: Soviet Phys.—JETP **14**, 1395 (1962)].

[10] M. Froissart, Phys. Rev. **123**, 1053 (1961).

make a satisfactory continuation in l satisfying the criteria of Carlson's theorem that it behaves asymptotically for large $|l|$ and for $\mathrm{Re}\,l>l_0$ as $A_l(s)=0(\exp\lambda|l|)$, $\lambda<\pi$.[11] This condition is necessary to insure the uniqueness of the continuation and to make possible the Watson-Sommerfeld transformation. The function δ_{l_0}, however, vanishes at all integers $l\geqslant 1$, and therefore has no continuation with the above bound. The custumary way to avoid this difficulty is to note that for $l\geqslant 1$ Eq. (8) becomes

$$A_l(s)=\frac{1}{2\pi\nu}\int_{t_0}^{\infty}dt' A_t(s,t')Q_l\left(1+\frac{t'}{2\nu}\right), \quad (9)$$

which now satisfies Carlson's criteria and may be used to define an analytic continuation $A(l,s)$ in $\mathrm{Re}\,l\geqslant 1$. This continuation agrees with the physical amplitude $A_l(s)$ for $l=1, 2, 3\cdots$. If no natural boundary exists, $A(l,s)$ can now be continued further to the left to the region of singularities. The difficulty now, however, is that there is no guarantee that this continuation $A(0,s)$ will give back the physical amplitude $A_0(s)$ at $l=0$, for any amplitude of the form

$$\bar{A}(s,t)=A(s,t)+f(s), \quad (10)$$

where $f(s)$ is an arbitrary function of s, will have the same projection for $l\geqslant 1$ as $A(s,t)$, and hence, the continuation down to $l=0$ is ambiguous.

If the nucleon is really a dynamical bound state of the π-N system then the pole term in the Mandelstam representation (7) should be omitted altogether, it should come automatically as a result of analytic continuation. This case corresponds to the discussion of the previous section, II. If on the other hand there is some additional contribution the continued function and the physical amplitude are then related by

$$A(0,s)=A_0(s)+f(s). \quad (11)$$

We now use the fact that both the physical and the continued amplitude satisfy the unitarity condition for $l=0$, the former by definition, the latter from the continuation of the unitarity. Because both $A(0,s)$ and $A_0(s)$ are required by this condition to be real below threshold, it follows immediately that

$$f(s) \text{ is real below threshold}. \quad (\mathrm{A})$$

In the elastic region the unitarity has the form

$$\mathrm{Im}A(0,s)=\lambda|A(0,s)|^2, \quad (12)$$

where λ is a constant depending on normalization. From (11), this gives

$$\mathrm{Im}[A_0(s)+f(s)]=\lambda|A_0(s)+f(s)|^2 \quad (13)$$

which, because $\mathrm{Im}A_0(s)=\lambda|A_0(s)|^2$, reduces to

$$\mathrm{Im}f(s)=\lambda\{[\mathrm{Re}f(s)]^2+[\mathrm{Im}f(s)]^2+2\,\mathrm{Re}A_0(s)\,\mathrm{Im}f(s)+2\,\mathrm{Im}A_0(s)\,\mathrm{Re}f(s)\}. \quad (14)$$

[11] E. C. Titchmarsh, *The Theory of Functions* (Oxford University Press, London, 1939), 2nd ed., p. 186.

Using the reality of $f(s)$ at threshold we obtain in the limit $\mathrm{Im}f(s)\xrightarrow[\nu\to 0+]{}0$

$$2\,\mathrm{Re}A_0(s)\,\mathrm{Re}f(s)+[\mathrm{Re}f(s)]^2=0,\ \nu\to 0_+. \quad (\mathrm{B})$$

We therefore conclude that $f(s)$ must be such that either

$$f(s)\to 0 \quad \text{as} \quad \nu\to 0_+ \quad (15)$$

or

$$f(s)\to -2\,\mathrm{Re}A_0(s) \quad \text{as} \quad \nu\to 0_+. \quad (16)$$

The first condition evidently comes from the possibility $f(s)\equiv 0$, i.e., that $A(0,s)$ and $A_0(s)$ are in fact the same. The second condition represents the possibility that these two amplitudes are not identical, and places an important restriction on $f(s)$ which allows one to approximate it near the threshold. Indeed, from (11) and (8), we have

$$f(s)=A(0,s)-\frac{g^2}{s-m^2}-C-\frac{1}{\pi}\int_{t_0}^{\infty}dt' A_t(s,t')\times\left[\frac{1}{2\nu}Q_l\left(1+\frac{t'}{2\nu}\right)-\frac{1}{t'}\right], \quad (17)$$

so that if the bound-state contribution $f_B=-g^2/(s-m^2)$ arising from the $\delta_{l_0}g^2/(s-m^2)$ term is dominant at threshold, one gets

$$f(s)\cong f_B(s)=-g^2/(s-m^2), \quad (18)$$

an assumption which can be checked by using Eq. (16) as we shall see.

Finally, we note from (16) that whenever $f(s)$ is real, one has

$$\mathrm{Re}A(0,s)=\pm\mathrm{Re}A_0(s), \quad \nu>0. \quad (\mathrm{C})$$

We will see that this relationship apparently depends only on the reality of $f(s)$ and is true even below threshold.

IV. PION-NUCLEON TRAJECTORY WITH AN "ELEMENTARY POLE"

We now apply the results of the previous section to the pion-nucleon case. In this case, the bound-state contribution to the amplitude $a_+^{1/2}(s)$ is given by

$$f_B(s)=[(W-m)^2-\mu^2]/4W^2\;3g^2/(W-m), \quad W=(s)^{1/2}. \quad (19)$$

The value of this term in the vicinity of the threshold is

$$f_B(s)\xrightarrow[\nu\to 0]{}0.210\nu, \quad (20)$$

in units with $\mu=1$, where we have used $f^2=0.08$, $g^2=(4m^2/\mu^2)f^2=14.43$. On the other hand Woolcook's value is

$$\mathrm{Re}(a_+^{1/2}(\nu)/\nu)\xrightarrow[\nu\to 0]{}-0.104\cong-\tfrac{1}{2}f_B(s). \quad (21)$$

Thus we see from (16) that the approximation (19) is quite good, and we therefore assume

$$f(s)=\frac{(W-m)^2-\mu^2}{4W^2}\frac{3g^2}{W-m} \quad (22)$$

in the energy range of calculations. We note however that this approximation cannot hold too far below and above the threshold: $f(s)$ may become eventually strongly complex above threshold and the nucleon Regge pole will give a strong contribution below threshold. If we now use the continuation

$$a(J=\tfrac{1}{2},s)=a^{1/2}(s)+f(s) \quad (23)$$

with $f(s)$ given by (22) and calculate the parameters of the trajectory as before, we find

$$\alpha(0)=0.37\,,\quad Y_J=2.59\,,\quad Y_\nu=-3.37\,,$$

with the residue

$$\bar{g}^2=-13.03\,.$$

By the condition (C) the residue of the physical amplitude is then

$$g^2=13.03\,.$$

The results are again plotted in Figs. 1 and 2. As can be seen the agreement at high energies is much improved, and in addition, the trend of the trajectory is in better agreement with what one would expect.

In conclusion the fit of the experimental data by a single trajectory with an elementary pole contribution seems to be much better than without such a term although the contribution of other singularities in the angular momentum plane, which are not known at present, may change this picture. In either case a good value of the coupling constant is obtained by this extrapolation. One should also note that we have used not directly experimental data but Woolcook's analysis of it which itself makes use of dispersion relations.